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Stochastic Processes and their Applications 117 (2007) 1234–1250

stochastic processes and their applications

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A local strict comparison theorem and converse comparison theorems for reflected backward stochastic differential equations[☆]

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Received 3 November 2005; received in revised form 9 September 2006; accepted 20 December 2006
Available online 16 January 2007

Abstract

A local strict comparison theorem and some converse comparison theorems are proved for reflected backward stochastic differential equations under suitable conditions.

MSC: 60H10; 60H30

Keywords: Reflected backward stochastic differential equations; Comparison theorem

1. Introduction

The comparison theorem turns out to be a classical result for backward stochastic differential equations (BSDEs). It allows us to compare the solutions of two real-valued BSDEs by

☆ Partially supported by the Natural Science Foundation of China under grants 10426022, 10325101 (Distinguished Youth Foundation, entitled "Control Theory of Stochastic Systems"), the Chang Jiang Scholars Programme, and the Science Foundation of the Chinese Ministry of Education under grant 20030246004.

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2006.12.008

comparing the terminal conditions and the generators. In a converse way, Peng is concerned in 1997 with the following converse comparison property for BSDEs: if the solutions of two real-valued BSDEs are equal at the initial time for any identical terminal condition, their generators are identical. For work on this problem, the reader is referred to among others: Chen [3], Briand et al. [1], Coquet et al. [2], and Jiang [9]. In their arguments, the strict comparison theorem for BSDEs plays a crucial role.

On the other hand, the solution Y of a reflected BSDE (RBSDE) characterizes the value process of an optimal stopping time problem, and the price process $\{Y_t\}_{0\leq t\leq T}$ of an American option is a solution of an RBSDE (see El Karoui et al. [5]):

$$Y_t = (X_T - k)^+ - \int_t^T [rY_s + \theta Z_s]\mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s, \tag{1.1}$$

with

$$Y_t \geq S_t := (X_t - k)^+, \quad \forall t \in [0, T]; \quad \int_0^T (Y_t - S_t)\mathrm{d}K_t = 0.$$

Here $\theta := \sigma^{-1}(\mu - r)$ is the premium of the market risk, and $\{X_t\}_{0\leq t\leq T}$ is the stock price process satisfying the following SDE:

$$X_t = X_0 + \int_0^t \mu X_s \mathrm{d}s + \int_0^t \sigma X_s \mathrm{d}B_s, \quad t \in [0, T].$$

Define the stopping time $\tau := \inf\{t : Y_t = S_t\}$, which is the time when the investor would take action to sell or buy the stock. The theory of RBSDEs existing in the literature only reveals how the price Y_t depends on the generator $g(y, z) := ry + \theta z$, $y \in \mathbb{R}$, $z \in \mathbb{R}$, and the strike price k (more generally speaking, the obstacle and the terminal value) as well. It is natural to ask how the premium θ (more generally speaking, the generator g) can be obtained from a family of American options parameterized by the strike price k. Then, the relations among the solution, the generator and the obstacle become interesting.

In this paper, we are concerned with comparison theorems and converse comparison theorems for RBSDEs under suitable conditions, which reveal some monotonicity between the solution, and the generator and the obstacle of a RBSDE.

The rest of the paper is organized as follows. In Section 2, we provide some preliminary results on BSDEs and RBSDEs. In Section 3, we first illustrate that, quite differently from BSDEs, RBSDEs do not have the global comparison property. Then we prove a local strict comparison theorem for RBSDEs. In Sections 4 and 5, we discuss converse comparison properties for RBSDEs when the obstacle is not previously given and when the obstacle is previously given, respectively. Some interesting comparison theorems are obtained in both cases.

2. Preliminaries

In this section, we give some basic results on BSDEs and RBSDEs. They will be used in the subsequent sections.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{B_t\}_{t\geq 0}$ be a d-dimensional standard Brownian motion on this space such that $B_0 = 0$. Denote by $\{\mathcal{F}_t\}_{t\geq 0}$ the filtration generated by the Brownian motion $\{B_t\}_{t\geq 0}$: $\mathcal{F}_t := \sigma\{B_s, s \in [0, t]\} \vee \mathcal{N}$, $t \in [0, T]$, where $\mathcal{N}$ is the set of all P-null subsets. Let $T > 0$ be a given real number. For any positive integer n and $z \in \mathbf{R}^n$, $|z|$ denotes the Euclidean norm.

Define the following two spaces of processes:

$$\mathcal{H}^2(0,T;\mathbf{R}^n) := \left\{ \{\psi_t\}_{0\le t\le T} \text{ is an } \mathbf{R}^n\text{-valued predictable process} \right. \\ \left. \text{s.t. } E\int_0^T |\psi_t|^2 \mathrm{d}t < +\infty \right\}$$

and

$$\mathcal{S}^2(0,T;\mathbf{R}) := \left\{ \{\psi_t\}_{0\le t\le T} \text{ is a predictable process s.t. } E\left[\sup_{0\le t\le T} |\psi_t|^2\right] < +\infty \right\}.$$

Consider the function $g : \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$ such that $\{g(t,y,z)\}_{t\in[0,T]}$ is progressively measurable for each (y,z) in $\mathbf{R}\times\mathbf{R}^d$. We make the following assumptions on g throughout the paper.

(A1) There exists a constant $K > 0$ such that a.s.,

$$|g(t,y_1,z_1) - g(t,y_2,z_2)| \le K(|y_1-y_2| + |z_1-z_2|), \quad \forall t \in [0,T], \\ \forall y_1, y_2 \in \mathbf{R}, z_1, z_2 \in \mathbf{R}^d.$$

(A2) The process $g(\cdot,0,0) \in \mathcal{H}^2(0,T;\mathbf{R})$.

(A3) $g(t,y,0) = 0$ a.s. for any $(t,y) \in [0,T]\times\mathbf{R}$.

(A4) The mapping $t \mapsto g(t,y,z)$ is continuous a.s. for any $(y,z) \in \mathbf{R}\times\mathbf{R}^d$.

Remark 2.1. It is obvious that Assumption (A3) implies Assumption (A2).

It is by now well known (see Pardoux and Peng [10] for the proof) that under Assumptions (A1) and (A2), for any random variable $\xi \in L^2(\Omega,\mathcal{F}_T,P)$, the BSDE

$$y_t = \xi + \int_t^T g(s,y_s,z_s)\mathrm{d}s - \int_t^T z_s\,\mathrm{d}B_s, \quad 0 \le t \le T \tag{2.1}$$

has a unique adapted solution $(y^{T,g,\xi}, z^{T,g,\xi}) \in \mathcal{S}^2(0,T;\mathbf{R}) \times \mathcal{H}^2(0,T;\mathbf{R}^d)$.

In the sequel, we always assume that g satisfies (A1) and (A2). We introduce the following operator $\varepsilon_{g,T}$: for any $\xi \in L^2(\Omega,\mathcal{F}_T,P)$, denote by $\varepsilon_{g,T}[\xi]$ and $\varepsilon_{g,T}[\xi|\mathcal{F}_t]$ the initial value $y_0^{T,g,\xi}$ and the value $y_t^{T,g,\xi}$ at time t of the solution to BSDE (2.1), respectively. For a stopping time τ, the operator $\varepsilon_{g,\tau}$ can be defined in an identical way.

We give some basic results of BSDEs, including Lemmas 2.1–2.3, which can be found in Briand et al. [1] or Peng [11], El Karoui et al. [6], and Jiang [9], respectively.

Lemma 2.1 (*Comparison Theorem*). *Assume that two fields g_1 and g_2 satisfy* (A1) *and* (A2). *Consider $\xi_1, \xi_2 \in L^2(\Omega,\mathcal{F}_T,P)$. We have*

(i) *(Monotonicity). If $\xi_1 \ge \xi_2$ and $g_1 \ge g_2$ a.s., then $\varepsilon_{g_1,T}[\xi_1] \ge \varepsilon_{g_2,T}[\xi_2]$, and $\varepsilon_{g_1,T}[\xi_1|\mathcal{F}_t] \ge \varepsilon_{g_2,T}[\xi_2|\mathcal{F}_t]$ a.s. for $t \in [0,T]$.*

(ii) *(Strict monotonicity). If $\xi_1 \ge \xi_2$ and $g_1 \ge g_2$ a.s., and $P(\{\xi_1 > \xi_2\}) > 0$, then $P(\{\varepsilon_{g,T}[\xi_1|\mathcal{F}_t] > \varepsilon_{g,T}[\xi_2|\mathcal{F}_t]\}) > 0$ for $t \in [0,T]$. In particular, $\varepsilon_{g,T}[\xi_1] > \varepsilon_{g,T}[\xi_2]$.*

Lemma 2.2. *Assume that the field g satisfies assumptions* (A1) *and* (A2). *Consider the stopping time $\tau \le T$ and $\xi \in L^2(\Omega,\mathcal{F}_\tau,P)$. Define*

$$\overline{g}(t,y,z) := g(t,y,z)1_{[0,\tau]}(t), \quad \forall (t,y,z) \in [0,T]\times\mathbf{R}\times\mathbf{R}^d.$$

Then

$$\varepsilon_{g,\tau}[\xi] = \varepsilon_{\overline{g},T}[\xi] \quad \text{and} \quad \varepsilon_{g,\tau}[\xi|\mathcal{F}_t] = \varepsilon_{\overline{g},T}[\xi|\mathcal{F}_t] \quad a.s.\ \text{for } t \in [0,\tau].$$

Lemma 2.3. *Assume that two functions g_1 and g_2 satisfy assumptions* (A1), (A2) *and* (A4). *Then the following two assertions are equivalent:*

(i) $\varepsilon_{g_1,\tau}[\xi] = \varepsilon_{g_2,\tau}[\xi]$ *for each stopping time $\tau \le T$ and any $\xi \in L^2(\Omega, \mathcal{F}_\tau, P)$.*
(ii) $g_1(t,y,z) = g_2(t,y,z)$ *a.s. for any $(t,y,z) \in [0,T] \times \mathbf{R} \times \mathbf{R}^d$.*

We introduce the conditional g-expectation (see Peng [11,12], Chen [3], Coquet et al. [2,4]). Suppose g satisfies (A1), (A3) and (A4). We set, for any stopping time τ taking values in $[0,T]$,

$$\varepsilon_g[\xi|\mathcal{F}_\tau] := y_\tau^{T,g,\xi} (= \varepsilon_{g,T}[\xi|\mathcal{F}_\tau]).$$

It can be shown that $\varepsilon_g[\xi|\mathcal{F}_\tau]$ is the unique $\mathcal{F}_\tau$-measurable, square-integrable random variable η such that

$$\varepsilon_g[1_A\eta] = \varepsilon_g[1_A\xi], \quad \forall A \in \mathcal{F}_\tau.$$

Therefore it is called the g-expectation conditioned on $\mathcal{F}_\tau$. Notice that the g-expectation ε_g is a particular example of the nonlinear expectation introduced in [3,4,11,12]. Now we borrow from [2] the converse comparison theorem for g-expectation.

Lemma 2.4. *Suppose that two functions g_1 and g_2 satisfy assumptions* (A1), (A3) *and* (A4). *Then the following two assertions are equivalent:*

(i) $\varepsilon_{g_1}[\xi] \ge \varepsilon_{g_2}[\xi]$ *for any $\xi \in L^2(\Omega, \mathcal{F}_T, P)$.*
(ii) $g_1(t,y,z) \ge g_2(t,y,z)$ *a.s. for any $(t,y,z) \in [0,T] \times \mathbf{R} \times \mathbf{R}^d$.*

A reflected BSDE is associated with a terminal condition $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, a generator g, and an "obstacle" process $\{S_t\}_{0\le t\le T}$. We make the following assumption:

(A5) $\{S_t\}_{0\le t\le T}$ is a continuous process such that $\{S_t\}_{0\le t\le T} \in \mathcal{S}^2(0,T;\mathbf{R})$.

The solution of a RBSDE is a triple (Y,Z,K) of $\mathcal{F}_t$-progressively measurable processes taking values in $\mathbf{R} \times \mathbf{R}^d \times \mathbf{R}_+$ and satisfying

(i) $Z \in \mathcal{H}^2(0,T;\mathbf{R}^d)$, $Y \in \mathcal{S}^2(0,T;\mathbf{R})$, and $K_T \in L^2(\Omega, \mathcal{F}_T, P)$;
(ii)

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s, \quad t \in [0,T]; \tag{2.2}$$

(iii) $Y_t \ge S_t$ a.s. for any $t \in [0,T]$;
(iv) $\{K_t\}$ is continuous and increasing, $K_0 = 0$ and $\int_0^T (Y_t - S_t)\mathrm{d}K_t = 0$.

The following two lemmas are borrowed from El Karoui et al. [7].

Lemma 2.5. *Assume that g satisfies* (A1) *and* (A2), *$\xi \in L^2(\Omega, \mathcal{F}_T, P)$, $\{S_t\}_{0\le t\le T}$ satisfies* (A5)*, and $S_T \le \xi$ a.s. Then RBSDE* (2.2) *has a unique solution (Y,Z,K).*

Remark 2.2. For simplicity, a given triple (ξ, g, S) is said to satisfy the Standard Assumptions if the generator g satisfies (A1) and (A2), the terminal value $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, the obstacle S satisfies (A5), and $S_T \le \xi$ a.s.

Lemma 2.6 (*Comparison Theorem*). *Suppose that two triples* (ξ_1, g_1, S^1) *and* (ξ_2, g_2, S^2) *satisfy the Standard Assumptions (in fact, it is sufficient for either* g_1 *or* g_2 *to satisfy the Lipschitz condition* (A1)*). Furthermore, we make the following assumptions:*

(i) $\xi_1 \le \xi_2$ *a.s.;*
(ii) $g_1(t, y, z) \le g_2(t, y, z)$ *a.s. for* $(t, y, z) \in [0, T] \times \mathbf{R} \times \mathbf{R}^d$*;*
(iii) $S_t^1 \le S_t^2$ *a.s. for* $t \in [0, T]$.

Let (Y^1, Z^1, K^1) *and* (Y^2, Z^2, K^2) *be adapted solutions of RBSDEs* (2.2) *with data* (ξ_1, g_1, S^1) *and* (ξ_2, g_2, S^2)*, respectively. Then* $Y_t^1 \le Y_t^2$ *a.s. for* $t \in [0, T]$.

Lemma 2.7. *Assume that* (ξ_1, g_1, S) *and* (ξ_2, g_2, S) *satisfy the Standard Assumptions. Furthermore, we make the following assumptions:*

(i) $\xi_1 \le \xi_2$ *a.s.;*
(ii) $g_1(t, y, z) \le g_2(t, y, z)$ *a.s. for* $(t, y, z) \in [0, T] \times \mathbf{R} \times \mathbf{R}^d$.

Let (Y^1, Z^1, K^1) *and* (Y^2, Z^2, K^2) *be adapted solutions of RBSDEs* (2.2) *with data* (ξ_1, g_1, S) *and* (ξ_2, g_2, S)*, respectively. Then we have*

$$K_t^1 \ge K_t^2 \quad a.s. \text{ for } t \in [0, T], \text{ and } K_t^1 - K_t^2 \text{ is increasing in time variable } t. \tag{2.3}$$

See Hamadène et al. [8, Proposition 41.3] for the detailed proof of Lemma 2.7.

3. Local strict comparison theorem for RBSDEs

In contrast to the case for BSDEs, the strict comparison theorem is not true in general for RBSDEs. Here are two counterexamples.

Example 3.1. Take $T = 1$, $g = \frac{1}{3}$, $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{1}{2}$, and $S_t = -2t + 1$ for $t \in [0, T]$. Then the solution (Y^1, Z^1, K^1) of RBSDE (2.2) with data (ξ_1, g, S) is given by

$$Y_t^1 = \begin{cases} 1 - 2t, & \text{if } 0 \le t \le \dfrac{1}{5} \\ \dfrac{2}{3} - \dfrac{1}{3}t, & \text{if } \dfrac{1}{5} < t \le 1; \end{cases} \qquad Z_t^1 = 0, \quad 0 \le t \le 1;$$

$$K_t^1 = \begin{cases} \dfrac{5}{3}t, & \text{if } 0 \le t \le \dfrac{1}{5} \\ \dfrac{1}{3}, & \text{if } \dfrac{1}{5} < t \le 1. \end{cases}$$

The solution (Y^2, Z^2, K^2) of RBSDE (2.2) with data (ξ_2, g, S) is given by

$$Y_t^2 = \begin{cases} 1 - 2t, & \text{if } 0 \le t \le \dfrac{1}{10} \\ \dfrac{5}{6} - \dfrac{1}{3}t, & \text{if } \dfrac{1}{10} < t \le 1; \end{cases} \qquad Z_t^2 = 0, \quad 0 \le t \le 1;$$

$$K_t^2 = \begin{cases} \dfrac{5}{3}t, & \text{if } 0 \le t \le \dfrac{1}{10} \\ \dfrac{1}{6}, & \text{if } \dfrac{1}{10} < t \le 1. \end{cases}$$

Obviously, $Y_t^1 = Y_t^2$ for $t \in [0, \frac{1}{10}]$ and $Y_t^1 < Y_t^2$ for $t \in (\frac{1}{10}, 1]$. Moreover, $K_t^1 \geq K_t^2$ for $t \in [0, 1]$. The strict comparison theorem for RBSDE (2.2) does not hold.

The following example shows that even if the generator is zero, it happens that the strict comparison theorem for RBSDE (2.2) may not be true.

Example 3.2. Take $T = 1$, $g = 0$, $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{1}{2}$, and $S_t = -2t + 1$ for $t \in [0, T]$. Then the solution (Y^1, Z^1, K^1) of RBSDE (2.2) with data (ξ_1, g, S) is given by

$$Y_t^1 = \begin{cases} 1 - 2t, & \text{if } 0 \leq t \leq \dfrac{1}{3} \\ \dfrac{1}{3}, & \text{if } \dfrac{1}{3} < t \leq 1; \end{cases} \qquad Z_t^1 = 0, \quad 0 \leq t \leq 1;$$

$$K_t^1 = \begin{cases} 2t, & \text{if } 0 \leq t \leq \dfrac{1}{3} \\ \dfrac{2}{3}, & \text{if } \dfrac{1}{3} < t \leq 1. \end{cases}$$

The solution (Y^2, Z^2, K^2) of RBSDE (2.2) with data (ξ_2, g, S) is given by

$$Y_t^2 = \begin{cases} 1 - 2t, & \text{if } 0 \leq t \leq \dfrac{1}{4} \\ \dfrac{1}{2}, & \text{if } \dfrac{1}{4} < t \leq 1; \end{cases} \qquad Z_t^2 = 0, \quad 0 \leq t \leq 1;$$

$$K_t^2 = \begin{cases} 2t, & \text{if } 0 \leq t \leq \dfrac{1}{4} \\ \dfrac{1}{2}, & \text{if } \dfrac{1}{4} < t \leq 1. \end{cases}$$

Obviously, $Y_t^1 = Y_t^2$ for $t \in [0, \frac{1}{4}]$ and $Y_t^1 < Y_t^2$ for $t \in (\frac{1}{4}, 1]$. Moreover, $K_t^1 \geq K_t^2$ for $t \in [0, 1]$. The strict comparison theorem for RBSDE (2.2) does not hold.

However, we have the local strict comparison theorem.

Theorem 3.1. *Suppose that two triples (ξ_1, g, S) and (ξ_2, g, S) satisfy the Standard Assumptions. Moreover, assume that*

$$\xi_1 \leq \xi_2 \quad a.s. \quad \textit{and} \quad P(\{\xi_1 < \xi_2\}) > 0.$$

Let (Y^1, Z^1, K^1) and (Y^2, Z^2, K^2) be adapted solutions of RBSDE (2.2) with data (ξ_1, g, S) and (ξ_2, g, S), respectively. Then there exists a stopping time τ such that $\tau < T$ almost surely and $P(\{Y_t^1 < Y_t^2,\ \forall t \in [\tau, T)\}) > 0$.

Proof. From Lemma 2.6, we have

$$Y_t^1 \leq Y_t^2 \quad \text{a.s. for } t \in [0, T]. \tag{3.1}$$

Note that Y^1 and Y^2 are continuous processes. Then Theorem 3.1 follows immediately from the following Lemma 3.1 on taking Y to be equal to $Y^2 - Y^1$. □

Lemma 3.1. *Let $\{Y_t, t \in [0, T]\}$ be an adapted nonnegative continuous process such that $P(\{Y_T > 0\}) > 0$. Then there exists a stopping time τ such that $\tau < T$ almost surely and $P(\{Y_t > 0,\ \forall t \in [\tau, T)\}) > 0$.*

Proof. Define a sequence of stopping times $\{\tau_k\}_{k=1}^{\infty}$ in the following way:

$$\tau_1 := 0$$

and

$$\tau_{k+1} = \inf\left\{t \geq \tau_k + \frac{1}{2}(T - \tau_k) : Y_t = 0\right\} \wedge T, \quad k = 1, 2, \ldots.$$

Note the convention that $\inf \varnothing = +\infty$. It is obvious that the sequence $\{\tau_k\}_{k=1}^{\infty}$ is both bounded by T and nondecreasing. Therefore, it has an almost sure limit τ, which is still a stopping time satisfying $\tau \leq T$. Since

$$\tau_{k+1} \geq \tau_k + \frac{1}{2}(T - \tau_k), \quad k = 1, 2, \ldots,$$

we have by passing to the limit that $\tau \geq \tau + \frac{1}{2}(T - \tau)$, that is, $\tau \geq T$. Hence, $\tau = T$.

Furthermore, we assert that there is some positive integer k_0 such that $P(\{\tau_{k_0} = T\}) > 0$. Otherwise, we have

$$P(\{\tau_k < T\}) = 1, \quad k = 1, 2, \ldots.$$

This implies the following:

$$Y_{\tau_k} = 0, \quad k = 1, 2, \ldots.$$

Passing to the limit, we have $Y_T = 0$ a.s., which contradicts the assumption that $P(\{Y_T > 0\}) > 0$.

Take the smallest integer $\tilde{k}$ among those positive integers k_0 such that $P(\{\tau_{k_0} = T\}) > 0$. Then, we have

$$\tilde{k} \geq 1, \quad P(\{\tau_{\tilde{k}} = T\}) > 0, \quad \tau_{\tilde{k}-1} < T \text{ a.s.}$$

We assert that the stopping time

$$\tilde{\tau} := \left[\tau_{\tilde{k}-1} + \frac{1}{2}(T - \tau_{\tilde{k}-1})\right] < T$$

is a desired one of the lemma. In fact, by definition of $\tau_{\tilde{k}}$, we have $Y_t > 0$ on the interval $[\tilde{\tau}, T)$ whenever $\tau_{\tilde{k}} = T$. Therefore,

$$P(\{Y_t > 0, \ \forall t \in [\tilde{\tau}, T)\}) \geq P(\{\tau_{\tilde{k}} = T\}) > 0. \tag{3.2}$$

The proof is complete. □

Corollary 3.1. *In Theorem 3.1, if g is either bounded from below by a nonnegative constant C_1 or satisfies* (A3), *ξ_i $(i = 1, 2)$ are bounded from below and the obstacle process S is bounded from above by a constant C_2, then we have*

$$Y_t^1 \leq Y_t^2 \ a.s., \quad and \quad P(\{Y_t^1 < Y_t^2\}) > 0 \quad for\ t \in [0, T].$$

In particular, we have $Y_0^1 < Y_0^2$.

Proof. Consider the following BSDEs:

$$Y_t'^i = \xi_i + \int_t^T g(s, Y_s'^i, Z_s'^i)\mathrm{d}s - \int_t^T Z_s'^i \mathrm{d}B_s, \quad 0 \leq t \leq T, \ i = 1, 2.$$

Obviously, it follows from Lemma 2.1 that $Y_t'^i \geq C_2 \geq S_t$ a.s., $i = 1, 2$, and $P(\{Y_t'^1 < Y_t'^2\}) > 0$ for $t \in [0, T]$. From Lemma 2.5, we have

$$Y_t^i = Y_t'^i, \qquad Z_t^i = Z_t'^i, \quad \text{and} \quad K_t^i = 0 \quad \text{a.s. for } t \in [0, T].$$

Therefore $P(\{Y_t^1 < Y_t^2\}) > 0$ for $t \in [0, T]$. Then the proof is complete. □

4. A converse problem for RBSDEs

In this section, we consider the general converse comparison theorem for RBSDE (2.2). Coquet et al. [2] prove the converse comparison for BSDEs: if g_1, g_2 satisfy (A1), (A3) and (A4), and $\varepsilon_{g_1,T}[\xi] \geq \varepsilon_{g_2,T}[\xi]$ for any $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, then $g_1(t, y, z) \geq g_2(t, y, z)$ a.s. for $(t, y, z) \in [0, T] \times \mathbf{R} \times \mathbf{R}^d$.

Consider the converse comparison for RBSDE (2.2). It is interesting since the strict comparison theorem is not true for RBSDE (2.2) and several arguments developed for BSDEs have to be modified for RBSDEs.

In the sequel, we always assume that the data (ξ, g, S) satisfies the Standard Assumptions for RBSDEs. We introduce the following operator ε_g^r: denote by $\varepsilon_{g,T}^r[\xi]$ and $\varepsilon_{g,T}^r[\xi|\mathcal{F}_t]$ the initial value $Y_0^{T,g,\xi,S}$ and the value $Y_t^{T,g,\xi,S}$ at time t of the solution of RBSDE (2.2) with data (ξ, g, S), respectively.

Theorem 4.1. *Suppose that two functions g_1 and g_2 satisfy assumptions* (A1), (A3) *and* (A4). *Then the following two conditions are equivalent:*

(i) $\varepsilon_{g_1,T}^r[\xi] \geq \varepsilon_{g_2,T}^r[\xi]$ *for any* $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ *and any obstacle process* $(S_t)_{0\leq t\leq T}$ *satisfying* (A5) *and* $\xi \geq S_T$ *a.s.*
(ii) $g_1(t, y, z) \geq g_2(t, y, z)$ *a.s. for* $(t, y, z) \in [0, T] \times \mathbf{R} \times \mathbf{R}^d$.

Proof. Thanks to Lemma 2.6, it is obvious that (ii) implies (i). It is sufficient to prove that (i) implies (ii).

For $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, consider the following BSDE:

$$\begin{cases} -\mathrm{d}y^i(t) = g_i(t, y^i(t), z^i(t))\mathrm{d}t - z^i(t)\mathrm{d}B_t, \quad t \in [0, T], \\ y^i(T) = \xi, \end{cases}$$

for $i = 1, 2$. In view of Lemma 2.4, it suffices to show that $\varepsilon_{g_1,T}[\xi] \geq \varepsilon_{g_2,T}[\xi]$.

Consider the following BSDE:

$$Y_t = \xi + \int_t^T [-K|Y_s| - K|Z_s|]\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s, \quad 0 \leq t \leq T,$$

where $K = \max(K_1, K_2)$ with K_1 and K_2 being the Lipschitz constants of g_1 and g_2, respectively. Then $\{Y_t\}_{0\leq t\leq T}$ satisfies (A5) (see El Karoui et al. [6] for the detailed proof). Set $S := Y$. Since $\xi \geq S_T$, from assumptions (A1), (A3) and Lemma 2.1, it follows that

$$y^i(t) \geq S_t \quad \text{a.s. for } t \in [0, T].$$

From Lemma 2.5, we see that $(y^i, z^i, 0)$ is the solution of RBSDE (2.2) with data (ξ, g_i, S) for $i = 1, 2$. In particular, $\varepsilon_{g_i,T}^r[\xi] = y^i(0)$ for $i = 1, 2$. Therefore, we get from the assumption that

$$\varepsilon_{g_1,T}[\xi] = \varepsilon_{g_1,T}^r[\xi] \geq \varepsilon_{g_2,T}^r[\xi] = \varepsilon_{g_2,T}[\xi].$$

The proof is complete. □

As an immediate consequence of Theorem 4.1, we have

Corollary 4.1. *Assume that functions g_1 and g_2 satisfy assumptions* (A1), (A3) *and* (A4). *Then the following two conditions are equivalent:*

(i) $\varepsilon^r_{g_1,T}[\xi] = \varepsilon^r_{g_2,T}[\xi]$ *for any* $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ *and any obstacle process* $\{S_t\}_{0\le t\le T}$ *satisfying* (A5) *and* $\xi \ge S_T$ *a.s.*

(ii) $g_1(t, y, z) = g_2(t, y, z)$ *a.s. for any* $(t, y, z) \in [0, T] \times \mathbf{R} \times \mathbf{R}^d$.

Remark 4.1. The assertion of Theorem 4.1 is not true if assumption (A3) fails to be satisfied. To show this fact, consider the following example:

$$g_1(t) = t1_{[0,\frac{T}{2})}(t) + \frac{T}{2}1_{[\frac{T}{2},T]}(t) \quad \text{and} \quad g_2(t) = \frac{T}{2}1_{[0,\frac{T}{2})}(t) + (T-t)1_{[\frac{T}{2},T]}(t).$$

Then, for any $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, any $\{S_t\}_{0\le t\le T}$ satisfying (A5), and $\xi \ge S_T$ a.s., it follows from El Karoui et al. [5, Proposition 2.3] that

$$\varepsilon^r_{g_1,T}[\xi] \le \varepsilon^r_{g_2,T}[\xi].$$

However, $g_1 \not\le g_2$.

Remark 4.2. If the obstacle process $\{S_t\}_{0\le t\le T}$ is previously given and $\varepsilon^r_{g_1,T}[\xi] \ge \varepsilon^r_{g_2,T}[\xi]$ only for those $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ such that $\xi \ge S_T$ a.s., then Theorem 4.1 is not true in general. It suffices to consider the following example:

$$g_1(t, y, z) = \mu_1(y - c_1)^- \wedge |z| \quad \text{and} \quad g_2(t, y, z) = \mu_2(y - c_2)^- \wedge |z|,$$

with $c_1 < c_2$ and $\mu_1 > \mu_2$. It is obvious that g_1 and g_2 satisfy assumptions (A1), (A3), and (A4). Furthermore, take $S_t = c_2,\ t \in [0, T]$. Then for any $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ satisfying $\xi \ge S_T$ a.s., we have

$$\varepsilon^r_{g_1,T}[\xi] = \varepsilon^r_{g_2,T}[\xi] \quad \text{and} \quad \varepsilon^r_{g_1,T}[\xi|\mathcal{F}_t] = \varepsilon^r_{g_2,T}[\xi|\mathcal{F}_t] \quad \text{a.s. for } t \in (0, T].$$

However, we have neither $g_1 \ge g_2$ nor $g_1 \le g_2$.

When assumption (A2) instead of (A3) is made on g, we have the following converse comparison result.

Theorem 4.2. *Assume that functions g_1 and g_2 satisfy assumptions* (A1), (A2) *and* (A4). *Then the following two conditions are equivalent:*

(i) $\varepsilon^r_{g_1,\tau}[\xi] = \varepsilon^r_{g_2,\tau}[\xi]$ *for any stopping time* $\tau \le T$, *any* $\xi \in L^2(\Omega, \mathcal{F}_\tau, P)$, *and any obstacle* $\{S_t\}_{0\le t\le T}$ *satisfying* (A5) *and* $\xi \ge S_\tau$ *a.s.*

(ii) $g_1(t, y, z) = g_2(t, y, z)$ *a.s. for any* $(t, y, z) \in [0, T] \times \mathbf{R} \times \mathbf{R}^d$.

Proof. Thanks to Lemma 2.6, it is obvious that (ii) implies (i). It is sufficient to prove that (i) implies (ii).

For $\xi \in L^2(\Omega, \mathcal{F}_\tau, P)$, consider the following BSDE defined on the interval $[0, \tau]$:

$$\begin{cases} -\mathrm{d}y^i(t) = g_i(t, y^i(t), z^i(t))\mathrm{d}t - z^i(t)\mathrm{d}B_t, \\ y^i(\tau) = \xi, \end{cases}$$

for $i = 1, 2$. In view of Lemma 2.3, it suffices to show that $\varepsilon_{g_1,\tau}[\xi] = \varepsilon_{g_2,\tau}[\xi]$.

Consider the obstacle process $\{S_t\}_{0\le t\le T}$ which is defined to be ξ on $[\tau, T]$, and on $[0, \tau]$ is taken to be one component of the solution of the following BSDE:

$$S_t = \xi + \int_t^\tau [g_1(s, 0, 0) \wedge g_2(s, 0, 0) - K|S_s| - K|Z_s|]\mathrm{d}s - \int_t^\tau Z_s \mathrm{d}B_s,$$

where $K = \max(K_1, K_2)$, K_1 and K_2 are the Lipschitz constants of g_1 and g_2, respectively. The above equation admits a unique solution $\{S_t\}_{0\le t\le T}$, which satisfies (A5). Since $\xi \ge S_\tau$, it follows from assumptions (A1), (A2), Lemmas 2.1 and 2.2 that

$$y^i(t) \ge S_t \quad \text{a.s. on the interval } [0, \tau].$$

From Lemma 2.5, we get that $(y^i, z^i, 0)$ is the solution of RBSDE (2.2) with data (ξ, g_i, S) on the interval $[0, \tau]$ for $i = 1, 2$. In particular, $\varepsilon^r_{g_i,T}[\xi] = y^i(0)$ for $i = 1, 2$. Therefore, it follows from the assumption that

$$\varepsilon_{g_1,\tau}[\xi] = \varepsilon^r_{g_1,\tau}[\xi] = \varepsilon^r_{g_2,\tau}[\xi] = \varepsilon_{g_2,\tau}[\xi].$$

The proof is complete. □

Remark 4.3. From the example in Remark 4.1, we can get that if g only satisfies (A1), (A2), and (A4), assertion (ii) of Theorem 4.2 fails to hold under the condition (i) of Corollary 4.1.

5. Alternative converse problem for RBSDEs with the obstacle process $\{S_t\}_{0\le t\le T}$ being given

Remark 4.2 shows that if the obstacle process $\{S_t\}_{0\le t\le T}$ is previously given, it is impossible in general to compare the generator g on the whole space $\Omega \times [0, T] \times \mathbf{R} \times \mathbf{R}^d$. In this section we shall show that we can still have the local converse comparison theorem for RBSDEs on an upper semi-space $\Omega \times [0, T] \times [C, +\infty) \times \mathbf{R}^d$, specified by the uniform upper bound C of the obstacle, which is actually the whole space if the generator does not depend on the first unknown variable y (see Theorem 5.2 below).

Assume that the data (ξ, g, S) satisfies the Standard Assumption for RBSDEs. But to emphasize the dependence on the obstacle process $\{S_t\}_{0\le t\le T}$, denote by $\varepsilon^{r,S}_{g,T}[\xi]$ and $\varepsilon^{r,S}_{g,T}[\xi|\mathcal{F}_t]$ the initial value $Y_0^{T,g,\xi,S}$ and the value $Y_t^{T,g,\xi,S}$ at time t of the solution of RBSDE (2.2) with data (ξ, g, S), respectively.

Proposition 5.1. *Suppose that g satisfies* (A1) *and* (A2), *and the obstacle process $\{S_t\}_{0\le t\le T}$ satisfies* (A5). *For the stopping time $\tau \le T$, and the terminal value $\xi \in L^2(\Omega, \mathcal{F}_\tau, P)$ such that $\xi \ge S_\tau$ a.s., then we have*

$$\varepsilon^{r,S}_{g,\tau}[\xi] = \varepsilon^{r,\overline{S}}_{\overline{g},T}[\xi]$$

where

$$\overline{g}(t, y, z) := g(t, y, z)1_{[0,\tau]}(t) \quad \textit{and} \quad \overline{S}_t := S_{t\wedge\tau} \quad \textit{a.s. for } (t, y, z) \in [0,\ T] \times \mathbf{R} \times \mathbf{R}^d.$$

Proof. Consider the solution $(Y^{\tau,g,\xi,S}, Z^{\tau,g,\xi,S}, K^{\tau,g,\xi,S})$ of RBSDE (2.2) with data (ξ, g, S) on the interval $[0, \tau]$, and $(\overline{Y}^{T,\overline{g},\xi,\overline{S}}, \overline{Z}^{T,\overline{g},\xi,\overline{S}}, \overline{K}^{T,\overline{g},\xi,\overline{S}})$ of RBSDE (2.2) with data $(\xi, \overline{g}, \overline{S})$ on the interval $[0, T]$. Obviously, $\varepsilon^{r,S}_{g,\tau}[\xi] = Y_0^{\tau,g,\xi,S}$ and $\varepsilon^{r,\overline{S}}_{\overline{g},T}[\xi] = \overline{Y}_0^{T,\overline{g},\xi,\overline{S}}$.

For simplicity, denote $(Y^{\tau,g,\xi,S}, Z^{\tau,g,\xi,S}, K^{\tau,g,\xi,S})$ and $(\overline{Y}^{T,\overline{g},\xi,\overline{S}}, \overline{Z}^{T,\overline{g},\xi,\overline{S}}, \overline{K}^{T,\overline{g},\xi,\overline{S}})$ by (Y, Z, K) and $(\overline{Y}, \overline{Z}, \overline{K})$, respectively. From Lemma 2.5, we have

$$\begin{aligned}&\overline{Y}(t) = \xi, \qquad \overline{Z}(t) = 0, \qquad \overline{K}(t) = \overline{K}(\tau) \quad \text{on the interval } (\tau, T];\\&\overline{Y}(t) = Y(t), \qquad \overline{Z}(t) = Z(t), \qquad \overline{K}(t) = K(t) \quad \text{on the interval } [0, \tau].\end{aligned}$$

Therefore,

$$\varepsilon^{r,S}_{g,\tau}[\xi] = \varepsilon^{r,\overline{S}}_{\overline{g},T}[\xi]. \quad \square$$

Theorem 5.1. *Assume that two functions g_1 and g_2 satisfy assumptions* (A1), (A3) *and* (A4)*, and the obstacle process $\{S_t\}_{0\le t\le T}$ satisfies* (A5). *Moreover, assume that there is a constant C such that*

$$\sup_{0\le t\le T} S_t \le C \quad a.s.$$

If for each stopping time $\tau \le T$, we have

$$\varepsilon^{r,S}_{g_1,\tau}[\xi] \ge \varepsilon^{r,S}_{g_2,\tau}[\xi] \quad \textit{for any } \xi \in L^2(\Omega, \mathcal{F}_\tau, P) \textit{ such that } \xi \ge S_\tau \ a.s.,$$

then we have

$$g_1(t, y, z) \ge g_2(t, y, z) \quad a.s. \textit{ for any } (t, y, z) \in [0, T] \times [C, +\infty) \times \mathbf{R}^d.$$

Proof. For each $\delta > 0$ and $(y, z) \in (C, +\infty) \times \mathbf{R}^d$, define the following stopping time:

$$\tau_\delta = \tau_\delta(y, z) = \inf\{t \ge 0 : g_1(t, y, z) \le g_2(t, y, z) - \delta\} \wedge T.$$

If the result does not hold, then there exists $\delta > 0$ and $(y, z) \in (C, +\infty) \times \mathbf{R}^d$ such that

$$P(\{\tau_\delta(y, z) < T\}) > 0.$$

For such a triple (δ, y, z), consider the following SDEs defined on the interval $[\tau_\delta, T]$:

$$\begin{cases}-\mathrm{d}Y^1(t) = g_1(t, Y^1(t), z)\mathrm{d}t - z\mathrm{d}B_t,\\ Y^1(\tau_\delta) = y\end{cases}$$

and

$$\begin{cases}-\mathrm{d}Y^2(t) = g_2(t, Y^2(t), z)\mathrm{d}t - z\mathrm{d}B_t,\\ Y^2(\tau_\delta) = y.\end{cases}$$

For $i = 1, 2$, the above equations admit a unique solution $Y^i \in \mathcal{S}^2(\tau_\delta, T; \mathbf{R})$.

Now we define the following stopping times:

$$\begin{aligned}\tau^1_\delta &= \inf\{t \ge \tau_\delta : Y^1_t \le S_t\} \wedge T,\\ \tau^2_\delta &= \inf\{t \ge \tau_\delta : Y^2_t \le S_t\} \wedge T,\\ \tau'_\delta &= \inf\left\{t \ge \tau_\delta : g_1(t, Y^1(t), z) \ge g_2(t, Y^2(t), z) - \frac{\delta}{2}\right\} \wedge T.\end{aligned}$$

Note that $\tau^1_\delta = \tau^2_\delta = \tau'_\delta = T$, if $\tau_\delta = T$. Obviously, $\{\tau_\delta < \tau^1_\delta\} = \{\tau_\delta < \tau^2_\delta\} = \{\tau_\delta < \tau'_\delta\} = \{\tau_\delta < T\}$. Define

$$\tau^3_\delta = \tau^1_\delta \wedge \tau^2_\delta \wedge \tau'_\delta.$$

Hence $P(\{\tau_\delta < \tau_\delta^3\}) > 0$. Moreover, we have $Y_t^1 > S_t$ and $Y_t^2 > S_t$ on the interval $[\tau_\delta, \tau_\delta^3)$. Then the solution $(Y^i(t), z, 0)$ is the solution of RSBDE (2.2) with data $(Y^i(\tau_\delta^3), g_i, S)$ on the interval $[\tau_\delta, \tau_\delta^3]$ for $i = 1, 2$.

We first get the following three lemmas.

Lemma 5.1. $\varepsilon^{r,S}_{g_i,\tau_\delta}[y] = \varepsilon_{g_i,\tau_\delta}[y] = y$ *for* $i = 1, 2$.

Proof. Consider the following BSDE defined on the interval $[0, \tau_\delta]$:

$$\begin{cases} -\mathrm{d}y^1(t) = g_1(t, y^1(t), z^1(t))\mathrm{d}t - z^1(t)\mathrm{d}B_t, \\ y^1(\tau_\delta) = y. \end{cases}$$

From assumption (A3), we see that

$$y^1(t) = y \quad \text{and} \quad z^1(t) = 0 \quad \text{on the interval } [0, \tau_\delta].$$

Obviously, the triple $(y^1, z^1, 0)$ is the solution of RSBDE (2.2) with data (y, g_1, S) on the interval $[0, \tau_\delta]$. Similarly, we have $\varepsilon^{r,S}_{g_2,\tau_\delta}[y] = \varepsilon_{g_2,\tau_\delta}[y] = y$. The proof is complete. □

Lemma 5.2. *The strict inequality* $Y^1(\tau_\delta^3) > Y^2(\tau_\delta^3)$ *holds on* $\{\tau_\delta < \tau_\delta^3\}$.

Proof. From the definitions of τ'_δ and Y^i, we have

$$Y^1(\tau_\delta^3) - Y^2(\tau_\delta^3) = \int_{\tau_\delta}^{\tau_\delta^3} [g_2(s, Y^2(s), z) - g_1(s, Y^1(s), z)]\mathrm{d}s \geq \frac{\delta}{2}(\tau_\delta^3 - \tau_\delta) > 0,$$

on $\{\tau_\delta < \tau_\delta^3\}$. The proof is complete. □

Lemma 5.3. $\varepsilon^{r,S}_{g_2,\tau_\delta^3}[Y^1(\tau_\delta^3)] = \varepsilon_{g_2,\tau_\delta^3}[Y^1(\tau_\delta^3)]$.

Proof. Consider the following BSDE:

$$\begin{cases} -\mathrm{d}\tilde{Y}^2(t) = g_2(t, \tilde{Y}^2(t), \tilde{Z}^2(t))\mathrm{d}t - \tilde{Z}^2(t)\mathrm{d}B_t, \quad t \in [0, \tau_\delta^3]; \\ \tilde{Y}^2(\tau_\delta^3) = Y^1(\tau_\delta^3). \end{cases}$$

From the definition of τ_δ^3 and Lemma 5.2, we get

$$Y^1(\tau_\delta^3) \geq Y^2(\tau_\delta^3) \quad \text{and} \quad P(\{Y^1(\tau_\delta^3) > Y^2(\tau_\delta^3)\}) > 0.$$

On the other hand, we have

$$\varepsilon^{r,S}_{g_2,\tau_\delta^3}[Y^2(\tau_\delta^3)] = \varepsilon_{g_2,\tau_\delta^3}[Y^2(\tau_\delta^3)] \quad \text{and} \quad \varepsilon^{r,S}_{g_2,\tau_\delta^3}[Y^2(\tau_\delta^3)|\mathcal{F}_t] = \varepsilon_{g_2,\tau_\delta^3}[Y^2(\tau_\delta^3)|\mathcal{F}_t]$$
$$\text{on } [0, \tau_\delta^3].$$

From Lemma 2.1, we get $\tilde{Y}^2(t) \geq \varepsilon_{g_2,\tau_\delta^3}[Y^2(\tau_\delta^3)|\mathcal{F}_t] \geq S(t)$ on $[0, \tau_\delta^3]$. Therefore $(\tilde{Y}^2, \tilde{Z}^2, 0)$ is the solution of RBSDE (2.2) with data $(Y^1(\tau_\delta^3), g_2, S)$ on the interval $[0, \tau_\delta^3]$. The proof is complete. □

Let us return to the proof of Theorem 5.1.

Thanks to Lemma 5.1, we have

$$y = \varepsilon^{r,S}_{g_1,\tau_\delta}[y] = \varepsilon^{r,S}_{g_1,\tau_\delta}[\varepsilon^{r,S}_{g_1,\tau_\delta^3}[Y^1(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}]] = \varepsilon^{r,S}_{g_1,\tau_\delta^3}[Y^1(\tau_\delta^3)],$$

and

$$y = \varepsilon^{r,S}_{g_2,\tau_\delta}[y] = \varepsilon^{r,S}_{g_2,\tau_\delta}[\varepsilon^{r,S}_{g_2,\tau^3_\delta}[Y^2(\tau^3_\delta)|\mathcal{F}_{\tau_\delta}]] = \varepsilon^{r,S}_{g_2,\tau^3_\delta}[Y^2(\tau^3_\delta)].$$

On the other hand, from the definition of τ^3_δ and Lemma 5.3, it follows that

$$\varepsilon^{r,S}_{g_2,\tau^3_\delta}[Y^2(\tau^3_\delta)] = \varepsilon_{g_2,\tau^3_\delta}[Y^2(\tau^3_\delta)]$$

and

$$\varepsilon^{r,S}_{g_2,\tau^3_\delta}[Y^1(\tau^3_\delta)] = \varepsilon_{g_2,\tau^3_\delta}[Y^1(\tau^3_\delta)],$$

respectively. Furthermore, from Lemma 2.2 we get

$$\varepsilon^{r,S}_{g_2,\tau^3_\delta}[Y^i(\tau^3_\delta)] = \varepsilon_{g_2,\tau^3_\delta}[Y^i(\tau^3_\delta)] = \varepsilon_{\overline{g}_2,T}[Y^i(\tau^3_\delta)], \quad i = 1, 2.$$

Here, $\overline{g}_2(t, y, z) := g_2(t, y, z)1_{[0,\tau^3_\delta]}(t)$ for a.e. $t \in [0, T]$ and any $(y, z) \in \mathbf{R} \times \mathbf{R}^d$. From the definition of τ^3_δ and Lemma 5.2, it follows that

$$Y^1(\tau^3_\delta) \geq Y^2(\tau^3_\delta) \quad \text{and} \quad P(\{Y^1(\tau^3_\delta) > Y^2(\tau^3_\delta)\}) > 0.$$

Therefore, in view of Lemma 2.1, we have

$$\varepsilon_{\overline{g}_2,T}[Y^2(\tau^3_\delta)] < \varepsilon_{\overline{g}_2,T}[Y^1(\tau^3_\delta)]. \tag{5.1}$$

Concluding the above, we get

$$\begin{aligned} y &= \varepsilon^{r,S}_{g_2,\tau^3_\delta}[Y^2(\tau^3_\delta)] = \varepsilon_{g_2,\tau^3_\delta}[Y^2(\tau^3_\delta)] = \varepsilon_{\overline{g}_2,T}[Y^2(\tau^3_\delta)] < \varepsilon_{\overline{g}_2,T}[Y^1(\tau^3_\delta)] = \varepsilon_{g_2,\tau^3_\delta}[Y^1(\tau^3_\delta)] \\ &= \varepsilon^{r,S}_{g_2,\tau^3_\delta}[Y^1(\tau^3_\delta)] \leq \varepsilon^{r,S}_{g_1,\tau^3_\delta}[Y^1(\tau^3_\delta)] = y. \end{aligned}$$

This is a contradiction. The proof is complete. □

Remark 5.1. Consider the example given in Remark 4.2. Furthermore, assume that $\mu_2 > 0$. Immediately, we have the following three facts: (i) $g_1(\cdot, y, \cdot) = g_2(\cdot, y, \cdot)$ when $y \geq c_2$; (ii) $g_1(\cdot, y, z) < g_2(\cdot, y, z)$ when $c_1 \leq y < c_2$ and $z \neq 0$; and (iii) $g_1(\cdot, y, z) > g_2(\cdot, y, z)$ when $y \leq c_1 - |z|$ and $z \neq 0$.

On the other hand, since

$$\varepsilon^r_{g_1,T}[\xi] = \varepsilon^r_{g_2,T}[\xi] \quad \text{and} \quad \varepsilon^r_{g_1,T}[\xi|\mathcal{F}_t] = \varepsilon^r_{g_2,T}[\xi|\mathcal{F}_t] \quad \text{a.s. for } t \in (0, T]$$

for any $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ satisfying $\xi \geq S_T$ a.s., we deduce the above fact (i) from Theorem 5.1. The other two facts (ii) and (iii) demonstrate that the conclusion of Theorem 5.1 is the best possible in the underlying example.

In Theorem 5.1, the bound assumption on the obstacle process appears to be very restrictive. In what follows, we show that if the generator of RBSDE (2.2) does not depend on the first unknown variable y, we can get the following global converse comparison result without the bound assumption.

Theorem 5.2. *Suppose that two fields g_1 and g_2 satisfy assumptions* (A1), (A3) *and* (A4)*, and the obstacle process $\{S_t\}_{0\leq t\leq T}$ satisfies* (A5). *Furthermore, assume that g_1 and g_2 do not depend on y. If for each stopping time $\tau \leq T$,*

$$\varepsilon^{r,S}_{g_1,\tau}[\xi] \geq \varepsilon^{r,S}_{g_2,\tau}[\xi] \quad \textit{for any } \xi \in L^2(\Omega, \mathcal{F}_\tau, P) \textit{ such that } \xi \geq S_\tau \textit{ a.s.}, \tag{5.2}$$

then we have

$$g_1(t,z) \geq g_2(t,z) \quad a.s. \text{ for } (t,z) \in [0,T] \times \mathbf{R}^d. \tag{5.3}$$

Proof. *Step* 1. If $\sup_{0\leq t\leq T} S_t$ is bounded from above, then the desired assertion is immediate.

Step 2. For a large integer n, define the stopping time

$$\tau_n := \inf\{t \geq 0 : S_t \geq n\} \wedge T.$$

Then $0 \leq \tau_n \leq T$ a.s. Since S_0 is a deterministic finite number and S is continuous, we have $\tau_n > 0$ a.s. for any $n > S_0 + 1$.

For every $n > S_0 + 1$, define $\overline{g}_i(t,z) := g_i(t,z)1_{[0,\tau_n]}(t)$ and $\overline{S}_t = S_{t\wedge\tau_n}$ for $(t,z) \in [0,T] \times \mathbf{R}^d$ with $i = 1, 2$. Then for each stopping time $\tau \leq T$ and any $\xi \in L^2(\Omega, \mathcal{F}_\tau, P)$ such that $\xi \geq \overline{S}_\tau$, if we have

$$\varepsilon^{r,\overline{S}}_{\overline{g}_1,\tau}[\xi] \geq \varepsilon^{r,\overline{S}}_{\overline{g}_2,\tau}[\xi], \tag{5.4}$$

then noting that $\overline{S}_t \leq n$ on $[0,T]$ (in view of the definition of τ_n), we have from Step 1 that

$$\overline{g}_1(t,z) \geq \overline{g}_2(t,z) \quad \text{a.s. for } (t,z) \in [0,T] \times \mathbf{R}^d.$$

That is,

$$g_1(t,z) \geq g_2(t,z) \quad \text{a.s. for } (t,z) \in [0,\tau_n] \times \mathbf{R}^d.$$

Obviously, $\tau_n \uparrow T$ as $n \to \infty$. Passing to limit, from assumption (A4) we get

$$g_1(t,z) \geq g_2(t,z) \quad \text{a.s. for } (t,z) \in [0,T] \times \mathbf{R}^d.$$

The proof is then complete. Therefore we only need to prove inequality (5.4).

Define $\tilde{g}_i(t,z) = \overline{g}_i(t,z)1_{[0,\tau]}(t)$ and $\tilde{S}_t = \overline{S}_{t\wedge\tau}$ for $(t,z) \in [0,T] \times \mathbf{R}^d$ with $i = 1, 2$. It follows that $\tilde{g}_i(t,z) = g_i(t,z)1_{[0,\tau\wedge\tau_n]}(t)$ and $\tilde{S}_t = S_{t\wedge\tau\wedge\tau_n}$ for $(t,z) \in [0,T] \times \mathbf{R}^d$, $i = 1, 2$. From Proposition 5.1, we have

$$\varepsilon^{r,\overline{S}}_{\overline{g}_i,\tau}[\xi] = \varepsilon^{r,\tilde{S}}_{\tilde{g}_i,T}[\xi], \quad i = 1, 2.$$

On the other hand, from the definitions of $\tilde{g}_1(t,z)$ and $\tilde{S}$, we have

$$\varepsilon^{r,\tilde{S}}_{\tilde{g}_1,T}[\xi] = \varepsilon^{r,\tilde{S}}_{\tilde{g}_1,\tau\wedge\tau_n}[\varepsilon^{r,\tilde{S}}_{\tilde{g}_1,T}[\xi|\mathcal{F}_{\tau\wedge\tau_n}]] = \varepsilon^{r,S}_{g_1,\tau\wedge\tau_n}[\varepsilon^{r,\tilde{S}}_{\tilde{g}_1,T}[\xi|\mathcal{F}_{\tau\wedge\tau_n}]].$$

Therefore

$$\varepsilon^{r,\overline{S}}_{\overline{g}_1,\tau}[\xi] = \varepsilon^{r,S}_{g_1,\tau\wedge\tau_n}[\varepsilon^{r,\tilde{S}}_{\tilde{g}_1,T}[\xi|\mathcal{F}_{\tau\wedge\tau_n}]].$$

Similarly,

$$\varepsilon^{r,\overline{S}}_{\overline{g}_2,\tau}[\xi] = \varepsilon^{r,S}_{g_2,\tau\wedge\tau_n}[\varepsilon^{r,\tilde{S}}_{\tilde{g}_2,T}[\xi|\mathcal{F}_{\tau\wedge\tau_n}]].$$

Also, thanks to the definitions of $\tilde{g}_1(t,z)$, $\tilde{g}_2(t,z)$ and $\tilde{S}$, we get

$$\varepsilon^{r,\tilde{S}}_{\tilde{g}_1,T}[\xi|\mathcal{F}_{\tau\wedge\tau_n}] = \varepsilon^{r,\tilde{S}}_{\tilde{g}_2,T}[\xi|\mathcal{F}_{\tau\wedge\tau_n}].$$

For simplicity, we set $\eta := \varepsilon^{r,\tilde{S}}_{\tilde{g}_1,T}[\xi|\mathcal{F}_{\tau\wedge\tau_n}]$. Obviously $\eta \in L^2(\Omega, \mathcal{F}_{\tau\wedge\tau_n}, P)$ and $\eta \geq S_{\tau\wedge\tau_n}$. Then from the assumption, it follows that

$$\varepsilon^{r,\overline{S}}_{\overline{g}_1,\tau}[\xi] = \varepsilon^{r,S}_{g_1,\tau\wedge\tau_n}[\eta] \geq \varepsilon^{r,S}_{g_2,\tau\wedge\tau_n}[\eta] = \varepsilon^{r,\overline{S}}_{\overline{g}_2,\tau}[\xi].$$

Now we end up with the proof. $\square$

Remark 5.2. Obviously, (5.2) and (5.3) in Theorem 5.2 are also equivalent.

If assumption (A3) is replaced with assumption (A2) in Theorem 5.1, then we have

Theorem 5.3. *Assume that two random fields g_1 and g_2 satisfy assumptions* (A1), (A2) *and* (A4), *and the obstacle process $\{S_t\}_{0\le t\le T}$ satisfies* (A5). *If for any two stopping times τ and σ such that $\tau \le \sigma \le T$,*

$$\varepsilon^{r,S}_{g_1,\sigma}[\xi|\mathcal{F}_\tau] \ge \varepsilon^{r,S}_{g_2,\sigma}[\xi|\mathcal{F}_\tau] \quad a.s. \textit{ for } \xi \in L^2(\Omega, \mathcal{F}_\sigma, P) \textit{ such that } \xi \ge S_\sigma \ a.s., \tag{5.5}$$

then for any continuous process $Y \in \mathcal{S}^2(0,T;\mathbf{R})$ such that $Y(t) \ge S_t$ a.s. with $t \in [0,T]$, we have

$$g_1(t, Y(t), z) \ge g_2(t, Y(t), z) \quad a.s. \textit{ for } (t,z) \in [0,T] \times \mathbf{R}^d. \tag{5.6}$$

In particular,

$$g_1(t, S(t), z) \ge g_2(t, S(t), z) \quad a.s. \textit{ for } (t,z) \in [0,T] \times \mathbf{R}^d. \tag{5.7}$$

Proof. In view of the continuity of $g_1(t,y,z)$ and $g_2(t,y,z)$ in y, it is sufficient to prove (5.6) for any continuous process $Y \in \mathcal{S}^2(0,T;\mathbf{R})$ such that $Y(t) \ge S_t + \epsilon$ a.s. with $t \in [0,T]$ for some constant $\epsilon > 0$. We shall prove it by contradiction.

Otherwise, there would exist $\delta > 0$ and $z \in \mathbf{R}^d$ such that

$$P(\{\tau_\delta(z) < T\}) > 0.$$

Here for $\delta > 0$ and $z \in \mathbf{R}^d$, we have defined the following stopping time:

$$\tau_\delta = \tau_\delta(z) := \inf\{t \ge 0 : g_1(t, Y(t), z) \le g_2(t, Y(t), z) - \delta\} \wedge T.$$

For such a pair (δ, z), analogous to the proof of Theorem 5.1, consider the following SDEs defined on the interval $[\tau_\delta, T]$:

$$\begin{cases} -\mathrm{d}Y^1(t) = g_1(t, Y^1(t), z)\mathrm{d}t - z\mathrm{d}B_t, \\ Y^1(\tau_\delta) = Y(\tau_\delta) \end{cases}$$

and

$$\begin{cases} -\mathrm{d}Y^2(t) = g_2(t, Y^2(t), z)\mathrm{d}t - z\mathrm{d}B_t, \\ Y^2(\tau_\delta) = Y(\tau_\delta). \end{cases}$$

The above SDEs admit unique solutions $Y^i \in \mathcal{S}^2(\tau_\delta, T; \mathbf{R})$ with $i = 1, 2$.

Define the following stopping times:

$$\tau_\delta^1 = \inf\{t \ge \tau_\delta : Y_t^1 \le S_t\} \wedge T,$$
$$\tau_\delta^2 = \inf\{t \ge \tau_\delta : Y_t^2 \le S_t\} \wedge T,$$

and

$$\tau_\delta' = \inf\left\{t \ge \tau_\delta : g_1(t, Y^1(t), z) \ge g_2(t, Y^2(t), z) - \frac{\delta}{2}\right\} \wedge T.$$

Note that $\tau_\delta^1 = \tau_\delta^2 = \tau_\delta' = T$, if $\tau_\delta = T$. Obviously, $\{\tau_\delta < \tau_\delta^1\} = \{\tau_\delta < \tau_\delta^2\} = \{\tau_\delta < \tau_\delta'\} = \{\tau_\delta < T\}$. We define

$$\tau_\delta^3 = \tau_\delta^1 \wedge \tau_\delta^2 \wedge \tau_\delta'.$$

Hence $P(\{\tau_\delta < \tau_\delta^3\}) > 0$. Moreover, we have $Y_t^1 > S_t$ and $Y_t^2 > S_t$ on the interval $[\tau_\delta, \tau_\delta^3)$. Therefore, the triple $(Y^i, z, 0)$ is the solution of RSBDE (2.2) with data $(Y^i(\tau_\delta^3), g_i, S)$ on the interval $[\tau_\delta, \tau_\delta^3]$ for $i = 1, 2$. Consequently,

$$\varepsilon^{r,S}_{g_1,\tau_\delta^3}[Y^1(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}] = \varepsilon_{g_1,\tau_\delta^3}[Y^1(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}] = Y(\tau_\delta)$$

and

$$\varepsilon^{r,S}_{g_2,\tau_\delta^3}[Y^2(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}] = \varepsilon_{g_2,\tau_\delta^3}[Y^2(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}] = Y(\tau_\delta).$$

Identically to the proof of Lemmas 5.2 and 5.3, we get

Lemma 5.4. *We have*

$$Y^1(\tau_\delta^3) > Y^2(\tau_\delta^3) \quad \text{on } \{\tau_\delta < \tau_\delta^3\} \tag{5.8}$$

and

$$\varepsilon^{r,S}_{g_2,\tau_\delta^3}[Y^1(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}] = \varepsilon_{g_2,\tau_\delta^3}[Y^1(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}]. \tag{5.9}$$

From the definition of τ_δ^3 and (5.8), we have

$$Y^1(\tau_\delta^3) \geq Y^2(\tau_\delta^3) \quad \text{a.s. and} \quad P(\{Y^1(\tau_\delta^3) > Y^2(\tau_\delta^3)\}) > 0.$$

Then it follows from Lemma 2.1 and (5.9) that

$$\begin{aligned}\varepsilon^{r,S}_{g_2,\tau_\delta^3}[Y^1(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}] &= \varepsilon_{g_2,\tau_\delta^3}[Y^1(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}] \geq \varepsilon_{g_2,\tau_\delta^3}[Y^2(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}] \\ &= \varepsilon^{r,S}_{g_2,\tau_\delta^3}[Y^2(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}] \quad \text{a.s.}\end{aligned}$$

and

$$\begin{aligned}&P(\{\varepsilon^{r,S}_{g_2,\tau_\delta^3}[Y^1(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}] > \varepsilon^{r,S}_{g_2,\tau_\delta^3}[Y^2(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}]\}) \\ &\quad = P(\{\varepsilon_{g_2,\tau_\delta^3}[Y^1(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}] > \varepsilon_{g_2,\tau_\delta^3}[Y^2(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}]\}) > 0.\end{aligned}$$

The last relation implies that

$$P(\{\varepsilon^{r,S}_{g_2,\tau_\delta^3}[Y^1(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}] > Y(\tau_\delta)\}) > 0$$

which contradicts the assumption that

$$\varepsilon^{r,S}_{g_2,\tau_\delta^3}[Y^1(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}] \leq \varepsilon^{r,S}_{g_1,\tau_\delta^3}[Y^1(\tau_\delta^3)|\mathcal{F}_{\tau_\delta}] = Y(\tau_\delta) \quad \text{a.s.}$$

The proof is complete. □

The following gives an immediate consequence of Theorem 5.3.

Corollary 5.1. *Suppose that two generators g_1, g_2 satisfy assumptions* (A1), (A2) *and* (A4), *and the obstacle process $\{S_t\}_{0\leq t\leq T}$ satisfies* (A5). *Furthermore, assume that g_1 and g_2 do not depend on the first unknown variable y. If for each pair of stopping times τ and σ such that $\tau \leq \sigma \leq T$, we have*

$$\varepsilon^{r,S}_{g_1,\sigma}[\xi|\mathcal{F}_\tau] \geq \varepsilon^{r,S}_{g_2,\sigma}[\xi|\mathcal{F}_\tau] \quad a.s. \text{ for any } \xi \in L^2(\Omega, \mathcal{F}_\sigma, P) \text{ such that } \xi \geq S_\sigma \ a.s.,$$

then we have

$$g_1(t,z) \geq g_2(t,z) \quad a.s.\ for\ any\ (t,z) \in [0,T] \times \mathbf{R}^d.$$

Acknowledgements

Both authors would like to thank Professor Shige Peng for his helpful comments, and Guangyan Jia and Zhiyong Yu for detecting an error in the original proof of Theorem 3.1. They are also very grateful to the Associate Editor and the anonymous referee for their helpful suggestions and criticisms.

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